Optimal weighted Hardy-Rellich inequalities on $H^2 \cap H_0^1$

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Abstract

We give necessary and sufficient conditions on a pair of positive radial functions V and W on a ball B of radius R in R^n, $n \geq 1$, so that the following inequalities hold

$$\int_B V(x)|\nabla u|^2 dx \geq \int_B W(x)u^2 dx + b \int_{\partial B} u^2 ds \ \text{ for all u} \in H^1(B),$$

and

$$\int_B V(x)|\Delta u|^2 dx \geq \int_B W(x)|\nabla u|^2 dx + b \int_{\partial B} |\nabla u|^2 ds \ \ \text{ for all u} \in H^2(B).$$

Then we present various classes of optimal weighted Hardy-Rellich inequalities on $H^2 \cap H_0^1$. The proofs are based on decomposition into spherical harmonics. These types inequalities are important in the study of fourth order elliptic equations with Navier boundary condition and systems of second order elliptic equations.

1 Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and $0 \in \Omega$. Let us recall that the classical Hardy-Rellich inequality assets that

$$\int_\Omega |\Delta u|^2 dx \geq \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} dx, \ \text{ for } \ u \in H_0^2(\Omega), \tag{1}$$

where the constant appearing in the above inequality is the best constant and it is never achieved in H_0^2. Recently there has been a flurry of activity about possible improvements of the following type

$$\text{If } n \geq 5 \ \text{ then } \ \int_\Omega |\Delta u|^2 dx - \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} dx \geq \int_\Omega W(x)u^2 dx \ \ \text{ for } u \in H_0^2(\Omega), \tag{2}$$

as well as

$$\text{If } n \geq 3 \ \text{ then } \ \int_\Omega |\Delta u|^2 dx - C(n) \int_\Omega \frac{|\nabla u|^2}{|x|^2} dx \geq \int_\Omega V(x)|\nabla u|^2 dx \ \ \text{ for all } u \in H_0^2(\Omega), \tag{3}$$

where V, W are certain explicit radially symmetric potentials of order lower than $\frac{1}{r^2}$ (for V) and $\frac{1}{r^4}$ (for W) (see [2], [3], [8], [10], [11], [15], and [18].
The inequality (1) was first proved by Rellich [17] for $u \in H_0^2(\Omega)$ and then it was extended to functions in $H^2(\Omega) \cap H_0^1(\Omega)$ by Donal et al. in [11]. So far most of the results about improved Hardy-Rellich inequalities and the inequalities of the form (3) are proved for $u \in H_0^2(\Omega)$ (see [8], [15], and [18]). The goal of this paper is to provide a general approach to prove optimal weighted Hardy-Rellich inequalities on $H^2(\Omega) \cap H_0^1(\Omega)$ and inequalities of type (3) on $H^2(\Omega)$ which are important in the study of fourth order elliptic equations with Navier boudary condition and systems of second order elliptic equations (see [16]).

We start – in section 2 – by giving necessary and sufficient conditions on positive radial functions V and W on a ball B in R^n, so that the following inequality holds for some $c > 0$ and $b < 0$:

$$\int_B V(x)|\nabla u|^2 dx \geq c \int_B W(x)u^2 dx + b \int_{\partial B} u^2 \text{ for all } u \in H^1(B). \tag{4}$$

Assuming that the ball B has radius R and that $\int_0^R \frac{1}{r^{n-1}V(r)}dr = +\infty$, the condition is simply that the ordinary differential equation

$$(\text{B}_{V,cW}) \qquad\qquad y''(r) + (\tfrac{n-1}{r} + \tfrac{V_r(r)}{V(r)})y'(r) + \tfrac{cW(r)}{V(r)}y(r) = 0$$

has a positive solution φ on the interval $(0, R)$ with $V(R)\frac{\varphi'(R)}{\varphi(R)} = b$. As in [15], we shall call such a couple (V, W) a *Bessel pair on* $(0, R)$. The *weight* of such a pair is then defined as

$$\beta(V, W; R) = \sup\big\{c;\, (\text{B}_{V,cW}) \text{ has a positive solution on } (0, R)\big\}. \tag{5}$$

We call W a Bessel potential if $(1, W)$ is a Bessel pair. This characterization makes an important connection between Hardy-type inequalities and the oscillatory behavior of the above equations. For a detailed analysis of Bessel pairs see [15]. The above theorem in the general form of improved Hardy-type inequalities which recently has been of interest for many authors (see [1], [4], [5], [6], [7], [9], [12], [13], [19], and [20]).

Here is the main result of this paper.

Theorem 1.1 *Let V and W be positive radial C^1-functions on $B\backslash\{0\}$, where B is a ball centered at zero with radius R in $\mathbb{R}^n$ ($n \geq 1$) such that $\int_0^R \frac{1}{r^{n-1}V(r)}dr = +\infty$ and $\int_0^R r^{n-1}V(r)dr < +\infty$. The following statements are then equivalent:*

1. *(V, W) is a Bessel pair on $(0, R)$ with $\theta := V(R)\frac{\varphi'(R)}{\varphi(R)}$, where φ is the corresponding solution of $(\text{B}_{(V,W)})$.*

2. *$\int_B V(x)|\nabla u|^2 dx \geq \int_B W(x)u^2 dx + \theta \int_{\partial B} u^2 ds$ for all $u \in C^\infty(\bar{B})$.*

3. *If $\lim_{r\to 0} r^\alpha V(r) = 0$ for some $\alpha < n - 2$, then the above are equivalent to*

   $$\int_B V(x)|\Delta u|^2 dx \geq \int_B W(x)|\nabla u|^2 dx + (n-1)\int_B (\tfrac{V(x)}{|x|^2} - \tfrac{V_r(|x|)}{|x|})|\nabla u|^2 dx + (\theta + (n-1)V(R))\int_{\partial B}|\nabla u|^2,$$

   for all radial $u \in C^\infty(\bar{B})$.

4. *If in addition, $W(r) - \frac{2V(r)}{r^2} + \frac{2V_r(r)}{r} - V_{rr}(r) \geq 0$ on $(0, R)$, then the above are equivalent to*

   $$\int_B V(x)|\Delta u|^2 dx \geq \int_B W(x)|\nabla u|^2 dx + (n-1)\int_B (\tfrac{V(x)}{|x|^2} - \tfrac{V_r(|x|)}{|x|})|\nabla u|^2 dx + (\theta + (n-1)V(R))\int_{\partial B}|\nabla u|^2,$$

   for all $u \in C^\infty(\bar{B})$.

Appropriate combinations of 4) and 2) in the above theorem and lead to a myriad of Hardy-Rellich type inequalities on $H^2(\Omega) \cap H_0^1(\Omega)$.

Remark 1.2 *The condition $W(r) - \frac{2V(r)}{r^2} + \frac{2V_r(r)}{r} - V_{rr}(r) \geq 0$ in the above theorem guarantees that the minimizing sequences are radial functions. We shall see in section 3 that even with out this condition our approach is applicable, although the minimizing sequences are no longer radial functions.*

Remark 1.3 *To see the importance and generality of the above theorem, notice that inequalities (7) and (8) in [16] which are the author's main tools to prove singularity of the extremal solutions in dimensions $n \geq 9$ (see [16]) are an immediate consequence of the above theorem combined with (4). This theorem will also allow us to extend most of the results about Hardy and Hardy-Rellich type inequalities on $C_0^\infty(\Omega)$ to corresponding inequalities on $C^\infty(\bar{\Omega})$ such as those in [15] and [18].*

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We shall show that for $-\frac{n}{2} \leq m \leq \frac{n-2}{2}$

$$H_{n,m} = \inf_{u \in H^2(B) \setminus \{0\}} \frac{\int_B \frac{|\Delta u|^2}{|x|^{2m}}}{\int_B \frac{|\nabla u|^2}{|x|^{2m+2}}} = \inf_{u \in H_0^2(B) \setminus \{0\}} \frac{\int_B \frac{|\Delta u|^2}{|x|^{2m}}}{\int_B \frac{|\nabla u|^2}{|x|^{2m+2}}}, \tag{6}$$

and for $-\frac{n}{2} \leq m \leq \frac{n-4}{2}$

$$a_{n,m} = \inf_{u \in H^2(B) \cap H_0^1(B) \setminus \{0\}} \frac{\int_B \frac{|\Delta u|^2}{|x|^{2m}}}{\int_B \frac{u^2}{|x|^{2m+4}}} = \frac{\int_B \frac{|\Delta u|^2}{|x|^{2m}}}{\int_B \frac{u^2}{|x|^{2m+4}}}, \tag{7}$$

where the constants $H_{n,m}$ and $a_{n,m}$ have been computed in [18] and then more generally in [15]. For example $a_{n,0} = \frac{n^2}{4}$ for $n \geq 5$, $a_{4,0} = 3$, and $a_{3,0} = \frac{25}{36}$.

The above general theorem also allows us to obtain improved Hardy-Rellich inequalities on $H^2(B) \cap H_0^1(B)$. For instance, assume W is a Bessel potential on $(0, R)$ and φ is the corresponding solution of $(B_{(1,W)})$ with $R \frac{\varphi'(R)}{\varphi(R)} \geq -\frac{n}{2}$. If $r \frac{W_r(r)}{W(r)}$ decreases to $-\lambda$ and $\lambda \leq n - 2$, then we have for all $H^2(B) \cap H_0^1(B)$

$$\int_B |\Delta u|^2 dx - \frac{n^2(n-4)^2}{16} \int_B \frac{u^2}{|x|^4} dx \geq \left(\frac{n^2}{4} + \frac{(n-\lambda-2)^2}{4}\right) \beta(W; R) \int_B \frac{W(x)}{|x|^2} u^2 dx. \tag{8}$$

By applying (8) to the various examples of Bessel functions, we can various improved Hardy-Rellich inequalities on $H^2(B) \cap H_0^1(B)$. Here are some basic examples of Bessel potentials, their corresponding solution φ of $(B_{(1,W)})$.

- $W \equiv 0$ is a Bessel potential on $(0, R)$ for any $R > 0$ and $\varphi = 1$.

- $W \equiv 1$ is a Bessel potential on $(0, R)$ for any $R > 0$, $\varphi(r) = J_0(\frac{\mu r}{R})$, where J_0 is the Bessel function and $z_0 = 2.4048...$ is the first zero of the Bessel function J_0. Moreover $R \frac{\varphi'(R)}{\varphi(R)} = -\frac{n}{2}$.

- For $k \geq 1$, $R > 0$, let $W_{k,\rho}(r) = \Sigma_{j=1}^k \frac{1}{r^2} \left(\prod_{i=1}^j \log^{(i)} \frac{\rho}{r}\right)^{-2}$ where the functions $\log^{(i)}$ are defined iteratively as follows: $\log^{(1)}(.) = \log(.)$ and for $k \geq 2$, $\log^{(k)}(.) = \log(\log^{(k-1)}(.))$. $W_{k,\rho}$ is then a Bessel potential on $(0, R)$ with the corresponding solution

$$\varphi_k = \left(\prod_{i=1}^j \log^{(i)} \frac{\rho}{r}\right)^{-\frac{1}{2}}.$$

  It is easy to see that for $\rho \geq R(e^{e^{e^{\cdot^{\cdot^{e((k-1)-times)}}}}})$ large enough we have $R \frac{\varphi_k'(R)}{\varphi_k(R)} \geq -\frac{n}{2}$.

- For $k \geq 1$, and $R > 0$, define $\tilde{W}_{k;\rho}(r) = \Sigma_{j=1}^k \frac{1}{r^2} X_1^2(\frac{r}{R}) X_2^2(\frac{r}{R}) \ldots X_{j-1}^2(\frac{r}{R}) X_j^2(\frac{r}{R})$ where the functions X_i are defined iteratively as follows: $X_1(t) = (1 - \log(t))^{-1}$ and for $k \geq 2$, $X_k(t) = X_1(X_{k-1}(t))$. Then again $\tilde{W}_{k,\rho}$ is a Bessel potential on $(0, R)$ with $\varphi_k = (X_1(\frac{r}{R}) X_2(\frac{r}{R}) \ldots X_{j-1}(\frac{r}{R}) X_k(\frac{r}{R}))^{\frac{1}{2}}$. Moreover, $R \frac{\varphi_k'(R)}{\varphi_k(R)} = -\frac{k}{2}$.

As an example, let $k \geq 1$ and choose $\rho \geq R(e^{e^{e^{\cdot^{\cdot^{e(k-times)}}}}})$ large enough so that $R \frac{\varphi'(R)}{\varphi(R)} \geq -\frac{n}{2}$, where

$$\varphi = \left(\prod_{i=1}^j \log^{(i)} \frac{\rho}{|x|}\right)^{\frac{1}{2}}. \tag{9}$$

Then we have

$$\int_B |\Delta u(x)|^2 dx \geq \frac{n^2(n-4)^2}{16} \int_B \frac{u^2}{|x|^4} dx + \left(1 + \frac{n(n-4)}{8}\right) \sum_{j=1}^k \int_B \frac{u^2}{|x|^4} \left(\prod_{i=1}^j \log^{(i)} \frac{\rho}{|x|}\right)^{-2} dx, \tag{10}$$

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for all $H^2(B) \cap H_0^1(B)$ which corresponds to the result od Adimurthi et al. [2].

More generally, we show that for any $-\frac{n}{2} \leq m < \frac{n-2}{2}$, and any W Bessel potential on a ball $B_R \subset R^n$ of radius R, if for the corresponding solution φ of $(B_{(1,W)})$ we have $R\frac{\varphi'(R)}{\varphi(R)} \geq -\frac{n}{2} - m$, then the following inequality holds for all $u \in C_0^\infty(B_R)$

$$\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} \geq a_{n,m} \int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx + \beta(W;R) \int_{B_R} W(x) \frac{|\nabla u|^2}{|x|^{2m}} dx. \qquad (11)$$

We also establish a more general version of equation (8). Assuming again that $\frac{rW'(r)}{W(r)}$ decreases to $-\lambda$ on $(0, R)$, and provided $m \leq \frac{n-4}{2}$ and $\frac{n}{2} + m \geq \lambda \geq n - 2m - 4$, we then have for all $u \in C_0^\infty(B_R)$,

$$\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} dx \geq \frac{(n+2m)^2(n-2m-4)^2}{16} \int_{B_R} \frac{u^2}{|x|^{2m+4}} dx$$

$$+ \beta(W;R)(\frac{(n+2m)^2}{4} + \frac{(n-2m-\lambda-2)^2}{4}) \int_{B_R} \frac{W(x)}{|x|^{2m+2}} u^2 dx. \qquad (12)$$

2   General Hardy Inequalities

Here is the main result of this section.

Theorem 2.1 *Let V and W be positive radial C^1-functions on $B_R \backslash \{0\}$, where B_R is a ball centered at zero with radius R ($0 < R \leq +\infty$) in $\mathbb{R}^n$ ($n \geq 1$). Assume that $\int_0^a \frac{1}{r^{n-1}V(r)} dr = +\infty$ and $\int_0^a r^{n-1}V(r)dr < \infty$ for some $0 < a < R$. Then the following two statements are equivalent:*

1. *The ordinary differential equation*

   $(B_{V,W})$ $\qquad\qquad\qquad\qquad y''(r) + (\frac{n-1}{r} + \frac{V_r(r)}{V(r)})y'(r) + \frac{W(r)}{V(r)}y(r) = 0$

   has a positive solution on the interval $(0, R]$ with $\theta := V(R)\frac{\varphi'(R)}{\varphi(R)}$.

2. *For all $u \in H^1(B_R)$*

   $(H_{V,W})$ $\qquad\qquad\qquad\qquad \int_{B_R} V(x)|\nabla u(x)|^2 dx \geq \int_{B_R} W(x)u^2 dx + \theta \int_{\partial B} u^2 \ ds.$

The above theorem allows to generalize all Hardy type inequalities on $H_0^1(\Omega)$ to a corresponding inequality on $H^1(\Omega)$. For instance we can get the following general form of the Caffarelli-Kohn-Nirenberg inequalities.

Corollary 2.2 *Assume B is the ball of radius R and and centered at zero in $\mathbb{R}^n$. If $a \leq \frac{n-2}{2}$, then*

$$\int_B |x|^{-2a}|\nabla u(x)|^2 dx \geq (\frac{n-2a-2}{2})^2 \int_B |x|^{-2a-2}u^2 dx - \frac{(n-2a-2)R^{-2a-1}}{2} \int_{\partial B} u^2 dx, \qquad (13)$$

for all $u \in H^1(B)$.

To prove Theorem 2.1 we shall need the following lemma.

Lemma 2.3 *Let V and W be positive radial C^1-functions on a ball $B \backslash \{0\}$, where B is a ball with radius R in $\mathbb{R}^n$ ($n \geq 1$) and centered at zero. Assume*

$$\int_B (V(x)|\nabla u|^2 - W(x)|u|^2) \, dx - \theta \int_{\partial B} u^2 ds \geq 0 \text{ for all } u \in H^1(B),$$

for some $\theta < 0$. Then there exists a C^2-supersolution to the following linear elliptic equation

$$-\operatorname{div}(V(x)\nabla u) - W(x)u = 0, \quad \text{in } B, \qquad (14)$$
$$u > 0 \quad \text{in } B \setminus \{0\}, \qquad (15)$$
$$V\nabla u.\nu = \theta u \quad \text{in } \partial B. \qquad (16)$$

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Proof: Define

$$\lambda_1(V) := \inf\left\{\frac{\int_B V(x)|\nabla\psi|^2 - W(x)|\psi|^2 - \theta\int_{\partial B} u^2}{\int_B |\psi|^2}; \ \ \psi \in C_0^\infty(B \setminus \{0\})\right\}.$$

By our assumption $\lambda_1(V) \geq 0$. Let (φ_n, λ_1^n) be the first eigenpair for the problem

$$
\begin{aligned}
(L - \lambda_1(V) - \lambda_1^n)\varphi_n &= 0 \ \ on \ \ B \setminus B_{\frac{R}{n}} \\
\varphi_n &= 0 \ \ on \ \ \partial B_{\frac{R}{n}} \\
V\nabla\varphi_n.\nu &= \theta\varphi_n \ \ on \ \ \partial B,
\end{aligned}
$$

where $Lu = -\text{div}(V(x)\nabla u) - W(x)u$, and $B_{\frac{R}{n}}$ is a ball of radius $\frac{R}{n}$, $n \geq 2$. The eigenfunctions can be chosen in such a way that $\varphi_n > 0$ on $B \setminus B_{\frac{R}{n}}$ and $\varphi_n(b) = 1$, for some $b \in B$ with $\frac{R}{2} < |b| < R$.
Note that $\lambda_1^n \downarrow 0$ as $n \to \infty$. Harnak's inequality yields that for any compact subset K, $\frac{\max_K \varphi_n}{\min_K \varphi_n} \leq C(K)$ with the later constant being independent of φ_n. Also standard elliptic estimates also yields that the family (φ_n) have also uniformly bounded derivatives on the compact sets $B - B_{\frac{R}{n}}$.
Therefore, there exists a subsequence $(\varphi_{n_{l_2}})_{l_2}$ of $(\varphi_n)_n$ such that $(\varphi_{n_{l_2}})_{l_2}$ converges to some $\varphi_2 \in C^2(B \setminus B(\frac{R}{2}))$. Now consider $(\varphi_{n_{l_2}})_{l_2}$ on $B \setminus B(\frac{R}{3})$. Again there exists a subsequence $(\varphi_{n_{l_3}})_{l_3}$ of $(\varphi_{n_{l_2}})_{l_2}$ which converges to $\varphi_3 \in C^2(B \setminus B(\frac{R}{3}))$, and $\varphi_3(x) = \varphi_2(x)$ for all $x \in B \setminus B(\frac{R}{2})$. By repeating this argument we get a supersolution $\varphi \in C^2(B \setminus \{0\})$ i.e. $L\varphi \geq 0$, such that $\varphi > 0$ on $B \setminus \{0\}$ and $V\nabla\varphi.\nu = \theta\varphi$ on ∂B. $\square$

Proof of Theorem 2.1: First we prove that 1) implies 2). Let $\varphi \in C^1(0, R]$ be a solution of $(B_{V,W})$ such that $\varphi(x) > 0$ for all $x \in (0, R)$. Define $\frac{u(x)}{\varphi(|x|)} = \psi(x)$. Then

$$|\nabla u|^2 = (\varphi'(|x|))^2\psi^2(x) + 2\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi + \varphi^2(|x|)|\nabla\psi|^2.$$

Hence,

$$V(|x|)|\nabla u|^2 \geq V(|x|)(\varphi'(|x|))^2\psi^2(x) + 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi(x).$$

Thus, we have

$$\int_B V(|x|)|\nabla u|^2 dx \geq \int_B V(|x|)(\varphi'(|x|))^2\psi^2(x)dx + \int_B 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi dx.$$

Let B_ϵ be a ball of radius ϵ centered at the origin. Integrate by parts to get

$$
\begin{aligned}
\int_B V(|x|)|\nabla u|^2 dx \geq & \int_B V(|x|)(\varphi'(|x|))^2\psi^2(x)dx + \int_{B_\epsilon} 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi dx \\
& + \int_{B\setminus B_\epsilon} 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi dx \\
= & \int_{B_\epsilon} V(|x|)(\varphi'(|x|))^2\psi^2(x)dx + \int_{B_\epsilon} 2V(|x|)\varphi'(|x|)\varphi(|x|)\psi(x)\frac{x}{|x|}.\nabla\psi dx \\
& - \int_{B\setminus B_\epsilon} \left\{\left(V(|x|)\varphi''(|x|)\varphi(|x|) + (\frac{(n-1)V(|x|)}{r} + V_r(|x|))\varphi'(|x|)\varphi(|x|)\right)\psi^2(x)\right\} dx \\
& + \int_{\partial(B\setminus B_\epsilon)} V(|x|)\varphi'(|x|)\varphi(|x|)\psi^2(x)ds
\end{aligned}
$$

Let $\epsilon \to 0$ and use Lemma 2.3 in [15] and the fact that φ is a solution of $(D_{v,w})$ to get

$$
\begin{aligned}
\int_B V(|x|)|\nabla u|^2 dx \geq & -\int_B [V(|x|)\varphi''(|x|) + (\frac{(n-1)V(|x|)}{r} + V_r(|x|))\varphi'(|x|)]\frac{u^2(x)}{\varphi(|x|)}dx \\
= & \int_B W(|x|)u^2(x)dx - \theta\int_{\partial B} u^2 ds.
\end{aligned}
$$

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To show that 2) implies 1), we assume that inequality $(H_{V,W})$ holds on a ball B of radius R, and then apply Lemma 2.3 to obtain a C^2-supersolution for the equation (14). Now take the surface average of u, that is

$$y(r) = \frac{1}{n\omega_w r^{n-1}} \int_{\partial B_r} u(x) dS = \frac{1}{n\omega_n} \int_{|\omega|=1} u(r\omega) d\omega > 0, \tag{17}$$

where ω_n denotes the volume of the unit ball in R^n. We may assume that the unit ball is contained in B (otherwise we just use a smaller ball). It is easy to see that $V(R)\frac{y'(R)}{y(R)} = \theta$. We clearly have

$$y''(r) + \frac{n-1}{r} y'(r) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r} \Delta u(x) dS. \tag{18}$$

Since $u(x)$ is a supersolution of (14), we have

$$\int_{\partial B_r} div(V(|x|)\nabla u) ds - \int_{\partial B} W(|x|) u dx \geq 0,$$

and therefore,

$$V(r) \int_{\partial B_r} \Delta u dS - V_r(r) \int_{\partial B_r} \nabla u.x ds - W(r) \int_{\partial B_r} u(x) ds \geq 0.$$

It follows that

$$V(r) \int_{\partial B_r} \Delta u dS - V_r(r) y'(r) - W(r) y(r) \geq 0, \tag{19}$$

and in view of (17), we see that y satisfies the inequality

$$V(r) y''(r) + (\frac{(n-1)V(r)}{r} + V_r(r)) y'(r) \leq -W(r) y(r), \quad for \quad 0 < r < R, \tag{20}$$

that is it is a positive supersolution y for $(B_{V,W})$ with $V(R)\frac{y'(R)}{y(R)} = \theta$. Standard results in ODE now allow us to conclude that $(B_{V,W})$ has actually a positive solution on $(0, R)$, and the proof of theorem 2.1 is now complete. $\qquad\square$

An immediate application of Theorem 2.6 in [15] and Theorem 2.1 is the following very general Hardy inequality.

Theorem 2.4 *Let $V(x) = V(|x|)$ be a strictly positive radial function on a smooth domain Ω containing 0 such that $R = \sup_{x\in\Omega} |x|$. Assume that for some $\lambda \in \mathbb{R}$*

$$\frac{rV_r(r)}{V(r)} + \lambda \geq 0 \ on \ (0,R) \ and \ \lim_{r\to 0} \frac{rV_r(r)}{V(r)} + \lambda = 0. \tag{21}$$

If $\lambda \leq n-2$, then the following inequality holds for any Bessel potential W on $(0, R)$:

$$\int_\Omega V(x)|\nabla u|^2 dx \geq (\frac{n-\lambda-2}{2})^2 \int_\Omega \frac{V(x)}{|x|^2} u^2 dx + \beta(W; R) \int_\Omega V(x) W(x) u^2 dx$$
$$+ V(R)(\frac{\varphi'(R)}{\varphi(R)} - \frac{n-\lambda-2}{2R}) \int_{\partial B} u^2 \quad for \ u \in H^1(\Omega),$$

where φ is the corresponding solution of $(B_{1,W})$.

Proof: Under our assumptions, it is easy to see that $y = r^{\frac{n-\lambda-2}{2}} \varphi(r)$ is a positive super-solution of $B_{(V,V(\frac{n-\lambda-2}{2})^2 r^{-2}+W)}$. Now apply Theorem 2.6 in [15] and Theorem 2.1 to complete the proof. $\qquad\square$

3 General Hardy-Rellich inequalities

Let $0 \in \Omega \subset R^n$ be a smooth domain, and denote

$$C_r^k(\bar{\Omega}) = \{v \in C^k(\bar{\Omega}) : \text{v is radial }\}.$$

We start by considering a general inequality for radial functions.

Theorem 3.1 *Let V and W be positive radial C^1-functions on a ball $B\backslash\{0\}$, where B is a ball with radius R in R^n ($n \geq 1$) and centered at zero. Assume $\int_0^R \frac{1}{r^{n-1}V(r)}dr = \infty$ and $\lim_{r \to 0} r^\alpha V(r) = 0$ for some $\alpha < n-2$. Then the following statements are equivalent:*

1. *(V,W) is a Bessel pair on $(0,R)$ with $\theta := V(R)\frac{\varphi'(R)}{\varphi(R)}$, where φ is the corresponding solution of $(B_{(V,W)})$.*

2. *If $\lim_{r \to 0} r^\alpha V(r) = 0$ for some $\alpha < n-2$, then the above are equivalent to*

   $$\int_B V(x)|\Delta u|^2 dx \geq \int_B W(x)|\nabla u|^2 dx + (n-1)\int_B (\tfrac{V(x)}{|x|^2} - \tfrac{V_r(|x|)}{|x|})|\nabla u|^2 dx + (\theta + (n-1)V(R))\int_{\partial B} |\nabla u|^2,$$

   for all radial $u \in C^\infty(\bar{B})$.

Proof: Assume $u \in C_r^\infty(\bar{B})$ and observe that

$$\int_B V(x)|\Delta u|^2 dx = n\omega_n\{\int_0^R V(r)u_{rr}^2 r^{n-1}dr + (n-1)^2\int_0^R V(r)\frac{u_r^2}{r^2}r^{n-1}dr + 2(n-1)\int_0^R V(r)uu_r r^{n-2}dr\}.$$

Setting $\nu = u_r$, we then have

$$\int_B V(x)|\Delta u|^2 dx = \int_B V(x)|\nabla \nu|^2 dx + (n-1)\int_B (\frac{V(|x|)}{|x|^2} - \frac{V_r(|x|)}{|x|})|\nu|^2 dx + (n-1)V(R)\int_{\partial B} |\nu|^2 ds.$$

Thus, $(\text{HR}_{V,W})$ for radial functions is equivalent to

$$\int_B V(x)|\nabla \nu|^2 dx \geq \int_B W(x)\nu^2 dx.$$

It therefore follows from Theorem 2.1 that 1) and 2) are equivalent. $\qquad\square$

3.1 The non-radial case

The decomposition of a function into its spherical harmonics will be one of our tools to prove our results. This idea has also been used in [18] and [15]. Let $u \in C^\infty(\bar{B})$. By decomposing u into spherical harmonics we get

$$u = \Sigma_{k=0}^\infty u_k \text{ where } u_k = f_k(|x|)\varphi_k(x)$$

and $(\varphi_k(x))_k$ are the orthonormal eigenfunctions of the Laplace-Beltrami operator with corresponding eigenvalues $c_k = k(N + k - 2)$, $k \geq 0$. The functions f_k belong to $u \in C^\infty([0, R])$, $f_k(R) = 0$, and satisfy $f_k(r) = O(r^k)$ and $f'(r) = O(r^{k-1})$ as $r \to 0$. In particular,

$$\varphi_0 = 1 \text{ and } f_0 = \tfrac{1}{n\omega_n r^{n-1}}\int_{\partial B_r} uds = \tfrac{1}{n\omega_n}\int_{|x|=1} u(rx)ds. \tag{22}$$

We also have for any $k \geq 0$, and any continuous real valued functions v and w on $(0, \infty)$,

$$\int_{R^n} V(|x|)|\Delta u_k|^2 dx = \int_{R^n} V(|x|)\big(\Delta f_k(|x|) - c_k\frac{f_k(|x|)}{|x|^2}\big)^2 dx, \tag{23}$$

and

$$\int_{R^n} W(|x|)|\nabla u_k|^2 dx = \int_{R^n} W(|x|)|\nabla f_k|^2 dx + c_k \int_{R^n} W(|x|)|x|^{-2}f_k^2 dx. \tag{24}$$

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Theorem 3.2 *Let V and W be positive radial C^1-functions on a ball $B \backslash \{0\}$, where B is a ball with radius R in $\mathbb{R}^n$ ($n \geq 1$) and centered at zero. Assume $\int_0^R \frac{1}{r^{n-1}V(r)} dr = \infty$ and $\lim_{r \to 0} r^\alpha V(r) = 0$ for some $\alpha < (n-2)$. If*

$$W(r) - \frac{2V(r)}{r^2} + \frac{2V_r(r)}{r} - V_{rr}(r) \geq 0 \quad for \quad 0 \leq r \leq R, \tag{25}$$

and the ordinary differential equation $(B_{V,W})$ has a positive solution φ on the interval $(0, R]$ such that

$$(n - 1 + R\frac{\varphi'(R)}{\varphi(R)})V(R) \geq 0, \tag{26}$$

then the following inequality holds for all $u \in H^2(B)$.

$$(\mathrm{HR}_{V,W}) \qquad \int_B V(x)|\Delta u|^2 dx \geq \int_B W(x)|\nabla u|^2 dx + (n-1)\int_B (\frac{V(x)}{|x|^2} - \frac{V_r(|x|)}{|x|})|\nabla u|^2 dx.$$

Moreover, if $\beta(V, W; R) \geq 1$, then the best constant is given by

$$\beta(V, W; R) = \sup \{c; \ (\mathrm{HR}_{V,cW}) \ holds\}. \tag{27}$$

Proof: Assume that the equation $(B_{V,W})$ has a positive solution on $(0, R]$. We prove that the inequality $(HR_{V,W})$ holds for all $u \in C_0^\infty(B)$ by frequently using that

$$\int_0^R V(r)|x'(r)|^2 r^{n-1} dr \geq \int_0^R W(r)x^2(r)r^{n-1} dr + V(R)\frac{\varphi'(R)}{\varphi(R)}R^{n-1}(x(R))^2 \text{ for all } x \in C^1(0, R]. \tag{28}$$

Indeed, for all $n \geq 1$ and $k \geq 0$ we have

$$\begin{aligned}
\frac{1}{nw_n}\int_{\mathbb{R}^n} V(x)|\Delta u_k|^2 dx &= \frac{1}{nw_n}\int_{\mathbb{R}^n} V(x)\big(\Delta f_k(|x|) - c_k\frac{f_k(|x|)}{|x|^2}\big)^2 dx \\
&= \int_0^R V(r)\big(f_k''(r) + \frac{n-1}{r}f_k'(r) - c_k\frac{f_k(r)}{r^2}\big)^2 r^{n-1} dr \\
&= \int_0^R V(r)(f_k''(r))^2 r^{n-1} dr + (n-1)^2\int_0^R V(r)(f_k'(r))^2 r^{n-3} dr \\
&\quad + c_k^2\int_0^R V(r)f_k^2(r)r^{n-5} + 2(n-1)\int_0^R V(r)f_k''(r)f_k'(r)r^{n-2} \\
&\quad - 2c_k\int_0^R V(r)f_k''(r)f_k(r)r^{n-3} dr - 2c_k(n-1)\int_0^R V(r)f_k'(r)f_k(r)r^{n-4} dr.
\end{aligned}$$

Integrate by parts and use (22) for $k = 0$ to get

$$\begin{aligned}
\frac{1}{n\omega_n}\int_{\mathbb{R}^n} V(x)|\Delta u_k|^2 dx &= \int_0^R V(r)(f_k''(r))^2 r^{n-1} dr + (n-1+2c_k)\int_0^R V(r)(f_k'(r))^2 r^{n-3} dr \tag{29} \\
&\quad + (2c_k(n-4) + c_k^2)\int_0^R V(r)r^{n-5}f_k^2(r) dr - (n-1)\int_0^R V_r(r)r^{n-2}(f_k')^2(r) dr \\
&\quad - c_k(n-5)\int_0^R V_r(r)f_k^2(r)r^{n-4} dr - c_k\int_0^R V_{rr}(r)f_k^2(r)r^{n-3} dr. \tag{30} \\
&\quad + (n-1)V(R)(f_k'(R))^2 R^{n-2}
\end{aligned}$$

Now define $g_k(r) = \frac{f_k(r)}{r}$ and note that $g_k(r) = O(r^{k-1})$ for all $k \geq 1$. We have

$$\begin{aligned}
\int_0^R V(r)(f_k'(r))^2 r^{n-3} &= \int_0^R V(r)(g_k'(r))^2 r^{n-1} dr + \int_0^R 2V(r)g_k(r)g_k'(r)r^{n-2} dr + \int_0^R V(r)g_k^2(r)r^{n-3} dr \\
&= \int_0^R V(r)(g_k'(r))^2 r^{n-1} dr - (n-3)\int_0^R V(r)g_k^2(r)r^{n-3} dr - \int_0^R V_r(r)g_k^2(r)r^{n-2} dr
\end{aligned}$$

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Thus,

$$\int_0^R V(r)(f_k'(r))^2 r^{n-3} \geq \int_0^R W(r)f_k^2(r)r^{n-3}dr - (n-3)\int_0^R V(r)f_k^2(r)r^{n-5}dr - \int_0^R V_r(r)f_k^2(r)r^{n-4}dr. \tag{31}$$

Substituting $2c_k \int_0^R V(r)(f_k'(r))^2 r^{n-3}$ in (29) by its lower estimate in the last inequality (31), we get

$$
\begin{aligned}
\frac{1}{n\omega_n}\int_{R^n} V(x)|\Delta u_k|^2 dx \quad \geq \quad & \int_0^R W(r)(f_k'(r))^2 r^{n-1}dr + \int_0^R W(r)(f_k(r))^2 r^{n-3}dr \\
+ \quad & (n-1)\int_0^R V(r)(f_k'(r))^2 r^{n-3}dr + c_k(n-1)\int_0^R V(r)(f_k(r))^2 r^{n-5}dr \\
- \quad & (n-1)\int_0^R V_r(r)r^{n-2}(f_k')^2(r)dr - c_k(n-1)\int_0^R V_r(r)r^{n-4}(f_k)^2(r)dr \\
+ \quad & c_k(c_k - (n-1))\int_0^R V(r)r^{n-5}f_k^2(r)dr \\
+ \quad & c_k\int_0^R \left(W(r) - \frac{2V(r)}{r^2} + \frac{2V_r(r)}{r} - V_{rr}(r)\right)f_k^2(r)r^{n-3}dr \\
+ \quad & (n-1)V(R)(f_k'(R))^2 R^{n-2} + V(R)\frac{\varphi'(R)}{\varphi(R)}R^{n-1}(f_k'(R))^2
\end{aligned}
$$

The proof is now complete since the last two terms are non-negative by our assumptions. $\qquad\square$

Remark 3.3 In order to apply the above theorem to

$$V(x) = |x|^{-2m}$$

we see that even in the simplest case $V \equiv 1$ condition (25) reduces to $(\frac{n-2}{2})^2|x|^{-2} \geq 2|x|^{-2}$, which is then guaranteed only if $n \geq 5$. More generally, if $V(x) = |x|^{-2m}$, then in order to satisfy (25) we need to have

$$\frac{-(n+4) - 2\sqrt{n^2 - n + 1}}{6} \leq m \leq \frac{-(n+4) + 2\sqrt{n^2 - n + 1}}{6}. \tag{32}$$

Also to satisfy the condition (26) we need to have $m > -\frac{n}{2}$. Thus for m satisfying (32) the inequality

$$\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} \geq (\frac{n+2m}{2})^2 \int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}}dx. \tag{33}$$

for all $u \in H^2(B_R)$. Moreover, $(\frac{n+2m}{2})^2$ is the best constant. We shall see however that this inequality remains true without condition (32), but with a constant that is sometimes different from $(\frac{n+2m}{2})^2$ in the cases where (32) is not valid. For example, if $m = 0$, then the best constant is 3 in dimension 4 and $\frac{25}{36}$ in dimension 3.

3.2 The case of power potentials $|x|^m$

The general Theorem 3.2 allowed us to deduce inequality (37) below for a restricted interval of powers m. We shall now prove that the same holds for all $-\frac{n}{2} \leq m < \frac{n-2}{2}$. We start with the following result.

Theorem 3.4 *Assume $-\frac{n}{2} \leq m < \frac{n-2}{2}$ and Ω be a smooth domain in $\mathbb{R}^n$, $n \geq 1$. Then*

$$a_{n,m} = \inf\left\{\frac{\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}}dx}{\int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}}dx}; H^2(\Omega) \setminus \{0\}\right\} = \inf\left\{\frac{\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}}dx}{\int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}}dx}; u \in H_0^2(\Omega) \setminus \{0\}\right\} \tag{34}$$

Proof. Decomposing again $u \in C^\infty(\bar{B}_R)$ into spherical harmonics; $u = \Sigma_{k=0}^\infty u_k$, where $u_k = f_k(|x|)\varphi_k(x)$, one has

$$\int_{\mathbb{R}^n} \frac{|\Delta u_k|^2}{|x|^{2m}} dx = \int_{\mathbb{R}^n} |x|^{-2m}(f_k''(|x|))^2 dx + ((n-1)(2m+1) + 2c_k)\int_{\mathbb{R}^n} |x|^{-2m-2}(f_k')^2 dx \qquad (35)$$

$$+ c_k(c_k + (n-4-2m)(2m+2))\int_{\mathbb{R}^n} |x|^{-2m-4}(f_k)^2 dx + (n-1)R^{n-2m-2}(f_k'(R))^2,$$

and

$$\int_{\mathbb{R}^n} \frac{|\nabla u_k|^2}{|x|^{2m+2}} dx = \int_{\mathbb{R}^n} |x|^{-2m-2}(f_k')^2 dx + c_k \int_{\mathbb{R}^n} |x|^{-2m-4}(f_k)^2 dx. \qquad (36)$$

The rest of the proof follows from the inequality (13) and an argument similar to that of Theorem 6.1 in [15]. $\qquad \square$

Remark 3.5 *The constant $a_{n,m}$ has been computed explicitly in [15] (Theorem 6.1).*

Theorem 3.6 *Suppose $n \geq 1$ and $-\frac{n}{2} \leq m < \frac{n-2}{2}$, and W is a Bessel potential on $B_R \subset R^n$ with $n \geq 3$ and φ is the corresponding solution for the $(B_{1,W})$. If*

$$R\frac{\varphi'(R)}{\varphi(R)} \geq -\frac{n}{2} - m,$$

then for all $u \in H^2(B_R)$ we have

$$\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} \geq a_{n,m} \int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx + \beta(W;R)\int_{B_R} W(x)\frac{|\nabla u|^2}{|x|^{2m}} dx, \qquad (37)$$

where

$$a_{n,m} = \inf \left\{ \frac{\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} dx}{\int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx}; u \in H^2(B_R) \setminus \{0\} \right\}.$$

Moreover $\beta(W;R)$ and $a_{m,n}$ are the best constants to be computed in the appendix.

Proof: Assuming the inequality

$$\int_{B_R} \frac{|\Delta u|^2}{|x|^{2m}} \geq a_{n,m} \int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx,$$

holds for all $u \in C^\infty(\bar{B}_R)$, we shall prove that it can be improved by any Bessel potential W. We will use the following inequality in the proof which follows directly from the inequality (13) with n=1.

$$\int_0^R r^\alpha (f'(r))^2 dr \geq (\frac{\alpha-1}{2})^2 \int_0^R r^{\alpha-2} f^2(r)dr + \beta(W;R)\int_0^R r^\alpha W(r)f^2(r)dr + (\frac{\varphi'(R)}{\varphi(R)} - \frac{\alpha-1}{2R})R^\alpha, \quad (38)$$

for $\alpha \geq 1$ and for all $f \in C^\infty(0,R]$, where both $(\frac{\alpha-1}{2})^2$ and $\beta(W;R)$ are best constants. Decompose $u \in C^\infty(\bar{B}_R)$ into its spherical harmonics $\Sigma_{k=0}^\infty u_k$, where $u_k = f_k(|x|)\varphi_k(x)$. We evaluate $I_k = \frac{1}{nw_n} \int_{R^n} \frac{|\Delta u_k|^2}{|x|^{2m}} dx$

in the following way

$$
\begin{aligned}
I_k &= \int_0^R r^{n-2m-1}(f_k''(r))^2 dr + [(n-1)(2m+1) + 2c_k]\int_0^R r^{n-2m-3}(f_k')^2 dr \\
&\quad + c_k[c_k + (n-2m-4)(2m+2)]\int_0^R r^{n-2m-5}(f_k(r))^2 dr \\
&\quad + (n-1)R^{n-2m-2}(f_k'(R))^2 \\
&\geq \beta(W)\int_0^R r^{n-2m-1}W(x)(f_k')^2 dr + [(\frac{n+2m}{2})^2 + 2c_k]\int_0^R r^{n-2m-3}(f_k')^2 dr \\
&\quad + c_k[c_k + (n-2m-4)(2m+2)]\int_0^R r^{n-2m-5}(f_k(r))^2 dr \\
&\geq \beta(W)\int_0^R r^{n-2m-1}W(x)(f_k')^2 dr + a_{n,m}\int_0^R r^{n-2m-3}(f_k')^2 dr \\
&\quad + \beta(W)[(\frac{n+2m}{2})^2 + 2c_k - a_{n,m}]\int_0^R r^{n-2m-3}W(x)(f_k)^2 dr \\
&\quad + ((\frac{n-2m-4}{2})^2[(\frac{n+2m}{2})^2 + 2c_k - a_{n,m}] + c_k[c_k + (n-2m-4)(2m+2)])\int_0^R r^{n-2m-5}(f_k(r))^2 dr.
\end{aligned}
$$

Now by (115) in [15] we have

$$
((\frac{n-2m-4}{2})^2[(\frac{n+2m}{2})^2 + 2c_k - a_{n,m}] + c_k[c_k + (n-2m-4)(2m+2)] \geq c_k a_{n,m},
$$

for all $k \geq 0$. Hence, we have

$$
\begin{aligned}
I_k &\geq a_{n,m}\int_0^R r^{n-2m-3}(f_k')^2 dr + a_{n,m}c_k\int_0^R r^{n-2m-5}(f_k(r))^2 dr \\
&\quad + \beta(W)\int_0^R r^{n-2m-1}W(x)(f_k')^2 dr + \beta(W)[(\frac{n+2m}{2})^2 + 2c_k - a_{n,m}]\int_0^R r^{n-2m-3}W(x)(f_k)^2 dr \\
&\geq a_{n,m}\int_0^R r^{n-2m-3}(f_k')^2 dr + a_{n,m}c_k\int_0^R r^{n-2m-5}(f_k(r))^2 dr \\
&\quad + \beta(W)\int_0^R r^{n-2m-1}W(x)(f_k')^2 dr + \beta(W)c_k\int_0^R r^{n-2m-3}W(x)(f_k)^2 dr \\
&= a_{n,m}\int_{B_R} \frac{|\nabla u|^2}{|x|^{2m+2}} dx + \beta(W)\int_{B_R} W(x)\frac{|\nabla u|^2}{|x|^{2m}} dx.
\end{aligned}
$$

$\square$

In the following theorem we prove a very general class of weighted Hardy-Rellich inequalities on $H^2(\Omega) \cap H_0^1$.

Theorem 3.7 *Let Ω be a smooth domain in R^n with $n \geq 1$ and let $V \in C^2(0, R =: \sup_{x \in \Omega}|x|)$ be a non-negative function that satisfies the following conditions:*

$$
V_r(r) \leq 0 \quad and \quad \int_0^R \frac{1}{r^{n-3}V(r)}dr = -\int_0^R \frac{1}{r^{n-4}V_r(r)}dr = +\infty. \tag{39}
$$

There exists $\lambda_1, \lambda_2 \in R$ such that

$$
\frac{rV_r(r)}{V(r)} + \lambda_1 \geq 0 \ on\ (0, R)\ and\ \lim_{r \to 0}\frac{rV_r(r)}{V(r)} + \lambda_1 = 0, \tag{40}
$$

$$
\frac{rV_{rr}(r)}{V_r(r)} + \lambda_2 \geq 0 \ on\ (0, R)\ and\ \lim_{r \to 0}\frac{rV_{rr}(r)}{V_r(r)} + \lambda_2 = 0, \tag{41}
$$

and

$$
\left(\tfrac{1}{2}(n - \lambda_1 - 2)^2 + 3(n-3)\right)V(r) - (n-5)rV_r(r) - r^2V_{rr}(r) \geq 0 \ for\ all\ r \in (0, R). \tag{42}
$$

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If $\lambda_1 \leq n$, then the following inequality holds:

$$\int_{\Omega} V(|x|)|\Delta u|^2 dx \geq (\frac{(n-\lambda_1-2)^2}{4} + (n-1))\frac{(n-\lambda_1-4)^2}{4}\int_{\Omega}\frac{V(|x|)}{|x|^4}u^2 dx$$

$$-\frac{(n-1)(n-\lambda_2-2)^2}{4}\int_{\Omega}\frac{V_r(|x|)}{|x|^3}u^2 dx. \tag{43}$$

Proof: We have by Theorem 2.4 and condition (42),

$$\frac{1}{n\omega_n}\int_{R^n}V(x)|\Delta u_k|^2 dx = \int_0^R V(r)(f_k''(r))^2 r^{n-1}dr + (n-1+2c_k)\int_0^R V(r)(f_k'(r))^2 r^{n-3}dr$$

$$+ (2c_k(n-4)+c_k^2)\int_0^R V(r)r^{n-5}f_k^2(r)dr - (n-1)\int_0^R V_r(r)r^{n-2}(f_k')^2(r)dr$$

$$- c_k(n-5)\int_0^R V_r(r)f_k^2(r)r^{n-4}dr - c_k\int_0^R V_{rr}(r)f_k^2(r)r^{n-3}dr$$

$$+ (n-1)V(R)(f_k'(R))^2 R^{n-2}$$

$$\geq \int_0^R V(r)(f_k''(r))^2 r^{n-1}dr + (n-1)\int_0^R V(r)(f_k'(r))^2 r^{n-3}dr$$

$$- (n-1)\int_0^R V_r(r)r^{n-2}(f_k')^2(r)dr$$

$$+ c_k\int_0^R \left(\left(\frac{1}{2}(n-\lambda_1-2)^2+3(n-3)\right)V(r)-(n-5)rV_r(r)-r^2 V_{rr}(r)\right)f_k^2(r)r^{n-5}dr$$

$$+ (n-1)V(R)(f_k'(R))^2 R^{n-2}$$

The rest of the proof follows from the above inequality combined with Theorem 2.4. $\square$

Remark 3.8 Let $V(r) = r^{-2m}$ with $-\frac{n}{2} \leq m \leq \frac{n-4}{2}$. Then in order to satisfy condition (42) we must have $-1-\frac{\sqrt{1+(n-1)^2}}{2} \leq m \leq \frac{n-4}{2}$. Since $-1-\frac{\sqrt{1+(n-1)^2}}{2} \leq -\frac{n}{2}$, if $-\frac{n}{2} \leq m \leq \frac{n-4}{2}$ the inequality (43) gives the following weighted second order Rellich inequality:

$$\int_B \frac{|\Delta u|^2}{|x|^{2m}}dx \geq H_{n,m}\int_B \frac{u^2}{|x|^{2m+4}}dx \quad u \in H^2(\Omega)\cap H_0^1(\Omega),$$

where

$$H_{n,m} := (\frac{(n+2m)(n-4-2m)}{4})^2. \tag{44}$$

The following theorem includes a large class of improved Hardy-Rellich inequalities as special cases.

Theorem 3.9 Let $-\frac{n}{2} \leq m \leq \frac{n-4}{2}$ and let $W(x)$ be a Bessel potential on a ball B of radius R in R^n with radius R. Assume $\frac{W(r)}{W_r(r)} = -\frac{\lambda}{r} + f(r)$, where $f(r) \geq 0$ and $\lim_{r\to 0} rf(r) = 0$. If $\lambda \leq \frac{n}{2}+m$, then the following inequality holds for all $u \in H^2 \cap H_0^1(B)$

$$\int_B \frac{|\Delta u|^2}{|x|^{2m}}dx \geq H_{n,m}\int_B \frac{u^2}{|x|^{2m+4}}dx$$

$$+\beta(W;R)(\frac{(n+2m)^2}{4}+\frac{(n-2m-\lambda-2)^2}{4})\int_B \frac{W(x)}{|x|^{2m+2}}u^2 dx. \tag{45}$$

Moreover, both constants are the best constants.

Proof: Again we will frequently use inequality (38) in the proof. Decomposing $u \in C^\infty(\bar{B}_R)$ into spherical harmonics $\Sigma_{k=0}^\infty u_k$, where $u_k = f_k(|x|)\varphi_k(x)$, we can write

$$
\begin{aligned}
\frac{1}{n\omega_n}\int_{R^n}\frac{|\Delta u_k|^2}{|x|^{2m}}dx \;=\; & \int_0^R r^{n-2m-1}(f_k''(r))^2 dr + [(n-1)(2m+1)+2c_k]\int_0^R r^{n-2m-3}(f_k')^2 dr \\
& + c_k[c_k + (n-2m-4)(2m+2)]\int_0^R r^{n-2m-5}(f_k(r))^2 dr \\
& + \; (n-1)(f_k'(R))^2 R^{n-2m-2} \\
\geq \;& (\frac{n+2m}{2})^2\int_0^R r^{n-2m-3}(f_k')^2 dr + \beta(W;R)\int_0^R r^{n-2m-1}W(x)(f_k')^2 dr \\
& + \; c_k[c_k + 2(\frac{n-\lambda-4}{2})^2 + (n-2m-4)(2m+2)]\int_0^R r^{n-2m-5}(f_k(r))^2 dr \\
& + \; (n-1)(f_k'(R))^2 R^{n-2m-2},
\end{aligned}
$$

where we have used the fact that $c_k \geq 0$ to get the above inequality. We have

$$
\begin{aligned}
\frac{1}{n\omega_n}\int_{R^n}\frac{|\Delta u_k|^2}{|x|^{2m}}dx \;\geq\; & \beta_{n,m}\int_0^R r^{n-2m-5}(f_k)^2 dr \\
& + \beta(W;R)\frac{(n+2m)^2}{4}\int_0^R r^{n-2m-3}W(x)(f_k)^2 dr \\
& + \beta(W;R)\int_0^R r^{n-2m-1}W(x)(f_k')^2 dr \\
\geq \;& \beta_{n,m}\int_0^R r^{n-2m-5}(f_k)^2 dr \\
& + \beta(W;R)(\frac{(n+2m)^2}{4} + \frac{(n-2m-\lambda-2)^2}{4})\int_0^R r^{n-2m-3}W(x)(f_k)^2 dr \\
\geq \;& \frac{\beta_{n,m}}{n\omega_n}\int_B \frac{u_k^2}{|x|^{2m+4}}dx \\
& + \frac{\beta(W;R)}{n\omega_n}(\frac{(n+2m)^2}{4} + \frac{(n-2m-\lambda-2)^2}{4})\int_B \frac{W(x)}{|x|^{2m+2}}u_k^2 dx,
\end{aligned}
$$

by Theorem 2.4. Hence, (45) holds and the proof is complete. $\qquad\square$

We shall now give a few immediate applications of the above in the case where $m=0$ and $n \geq 3$.

Theorem 3.10 *Assume W is a Bessel potential on $B_R \subset R^n$ with $n \geq 3$ and φ is the corresponding solution for the $(B_{1,W})$. If*

$$
R\frac{\varphi'(R)}{\varphi(R)} \geq -\frac{n}{2},
$$

then for all $u \in H^2(B_R)$ we have

$$
\int_{B_R}|\Delta u|^2 dx \geq C(n)\int_{B_R}\frac{|\nabla u|^2}{|x|^2}dx + \beta(W;R)\int_{B_R}W(x)|\nabla u|^2 dx, \tag{46}
$$

where $C(3)=\frac{25}{36}$, $C(4)=3$ and $C(n)=\frac{n^2}{4}$ for all $n \geq 5$. Moreover, $C(n)$ and $\beta(W;R)$ are best constants.

Corollary 3.11 *The following holds for any smooth bounded domain Ω in R^n with $R = \sup_{x\in\Omega}|x|$, and any $u \in H^2(\Omega)$.*

1. Let z_0 be the first zero of the Bessel function $J_0(z)$ and choose $0 < \mu < z_0$ so that

$$\mu \frac{J_0'(\mu)}{J_0(\mu)} = -\frac{n}{2}. \tag{47}$$

Then

$$\int_\Omega |\Delta u|^2 dx \geq C(n) \int_\Omega \frac{|\nabla u|^2}{|x|^2} dx + \frac{\mu^2}{R^2} \int_\Omega |\nabla u|^2 dx \tag{48}$$

2. For any $k \geq 1$, choose $\rho \geq R(e^{e^{e^{\cdot^{\cdot^{e(k-times)}}}}})$ large enough so that $R\frac{\varphi'(R)}{\varphi(R)} \geq -\frac{n}{2}$, where

$$\varphi = \left(\prod_{i=1}^{j} log^{(i)} \frac{\rho}{|x|} \right)^{\frac{1}{2}}. \tag{49}$$

Then we have

$$\int_\Omega |\Delta u(x)|^2 dx \geq C(n) \int_\Omega \frac{|\nabla u|^2}{|x|^2} dx + \frac{1}{4} \sum_{j=1}^{k} \int_\Omega \frac{|\nabla u|^2}{|x|^2} \left(\prod_{i=1}^{j} log^{(i)} \frac{\rho}{|x|} \right)^{-2} dx, \tag{50}$$

3. We have

$$\int_\Omega |\Delta u(x)|^2 dx \geq C(n) \int_\Omega \frac{|\nabla u|^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^{n} \int_\Omega \frac{|\nabla u|}{|x|^2} X_1^2(\frac{|x|}{R}) X_2^2(\frac{|x|}{R})...X_i^2(\frac{|x|}{R}) dx. \tag{51}$$

The following is immediate from Theorem 3.9 and from the fact that $\lambda = 2$ for the Bessel potential under consideration.

Corollary 3.12 *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 4$ and $R = \sup_{x \in \Omega} |x|$. Then the following holds for all $u \in H^2(\Omega) \cap H_0^1(\Omega)$*

1. Choose $\rho \geq R(e^{e^{e^{\cdot^{\cdot^{e(k-times)}}}}})$ so that $R\frac{\varphi'(R)}{\varphi(R)} \geq -\frac{n}{2}$. Then

$$\int_\Omega |\Delta u(x)|^2 dx \geq \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} dx + (1 + \frac{n(n-4)}{8}) \sum_{j=1}^{k} \int_\Omega \frac{u^2}{|x|^4} \left(\prod_{i=1}^{j} log^{(i)} \frac{\rho}{|x|} \right)^{-2} dx. \tag{52}$$

2. Let X_i is defined as in the introduction, then

$$\int_\Omega |\Delta u(x)|^2 dx \geq \frac{n^2(n-4)^2}{16} \int_\Omega \frac{u^2}{|x|^4} dx + (1 + \frac{n(n-4)}{8}) \sum_{i=1}^{n} \int_\Omega \frac{u^2}{|x|^4} X_1^2(\frac{|x|}{R}) X_2^2(\frac{|x|}{R})...X_i^2(\frac{|x|}{R}) dx. \tag{53}$$

Moreover, all constants in the above inequalities are best constants.

Theorem 3.13 *Let $W_1(x)$ and $W_2(x)$ be two radial Bessel potentials on a ball B of radius R in R^n with $n \geq 4$. Then for all $u \in H^2(B) \cap H_0^1(B)$*

$$\begin{aligned}
\int_B |\Delta u|^2 dx &\geq \frac{n^2(n-4)^2}{16} \int_B \frac{u^2}{|x|^4} dx + \frac{n^2}{4} \beta(W_1; R) \int_B W_1(x) \frac{u^2}{|x|^2} dx \\
&\quad + \mu(\frac{n-2}{2})^2 \int_B \frac{u^2}{|x|^2} dx + \mu\beta(W_2; R) \int_B W_2(x) u^2 dx,
\end{aligned}$$

where μ is defined by (47).

Proof: Here again we shall give a proof when $n \geq 5$. The case $n = 4$ will be handled in the next section. We again first use Theorem 3.10 (for $n \geq 5$), then Theorem 2.15 in [15] with the Bessel pair $(|x|^{-2}, |x|^{-2}(\frac{(n-4)^2}{4}|x|^{-2} + W))$, then again Theorem 2.1 with the Bessel pair $(1, (\frac{n-2}{2})^2|x|^{-2} + W)$ to obtain

$$
\begin{aligned}
\int_B |\Delta u|^2 dx &\geq \frac{n^2}{4} \int_B \frac{|\nabla u|^2}{|x|^2} dx + \mu \int_B |\nabla u|^2 dx \\
&\geq \frac{n^2(n-4)^2}{16} \int_B \frac{u^2}{|x|^4} dx + \frac{n^2}{4}\beta(W_1; R) \int_B W_1(x)\frac{u^2}{|x|^2} dx + \mu \int_B |\nabla u|^2 dx dx \\
&\geq \frac{n^2(n-4)^2}{16} \int_B \frac{u^2}{|x|^4} dx + \frac{n^2}{4}\beta(W_1; R) \int_B W_1(x)\frac{u^2}{|x|^2} dx \\
&\quad + \mu(\frac{n-2}{2})^2 \int_B \frac{u^2}{|x|^2} dx + \mu\beta(W_2; R) \int_B W_2(x)u^2 dx.
\end{aligned}
$$

Theorem 3.14 *Assume $n \geq 4$ and let $W(x)$ be a Bessel potential on a ball B of radius R and centered at zero in R^n. Then the following holds for all $u \in H^2(B) \cap H_0^1(B)$:*

$$
\int_B |\Delta u|^2 dx \geq \frac{n^2(n-4)^2}{16} \int_B \frac{u^2}{|x|^4} dx \tag{54}
$$

$$
+ \beta(W; R)\frac{n^2}{4} \int_B \frac{W(x)}{|x|^2} u^2 dx + \frac{\mu^2}{R^2}||u||_{H_0^1}, \tag{55}
$$

where $\frac{\mu^2}{R^2}$ is defined by (47).

Proof: Decomposing again $u \in C^\infty(\bar{B}_R)$ into its spherical harmonics $\Sigma_{k=0}^\infty u_k$ where $u_k = f_k(|x|)\varphi_k(x)$, we calculate

$$
\begin{aligned}
\frac{1}{n\omega_n} \int_{R^n} |\Delta u_k|^2 dx &= \int_0^R r^{n-1}(f_k''(r))^2 dr + [n-1+2c_k]\int_0^R r^{n-3}(f_k')^2 dr \\
&\quad + c_k[c_k + n - 4]\int_0^R r^{n-5}(f_k(r))^2 dr \\
&\quad + (n-1)(f_k'(R))^2 R^{n-2m-2} \\
&\geq \frac{n^2}{4}\int_0^R r^{n-3}(f_k')^2 dr + \frac{\mu^2}{R^2}\int_0^R r^{n-1}(f_k')^2 dr \\
&\quad + c_k \int_0^R r^{n-3}(f_k')^2 dr \\
&\geq \frac{n^2(n-4)^2}{16}\int_0^R r^{n-5}(f_k)^2 dr \\
&\quad + \beta(W; R)\frac{n^2}{4}\int_0^R W(r)r^{n-3}(f_k)^2 dr \\
&\quad + \frac{\mu^2}{R^2}\int_0^R r^{n-1}(f_k')^2 dr + c_k\frac{\mu^2}{R^2}\int_0^R r^{n-3}(f_k)^2 dr \\
&= \frac{n^2(n-4)^2}{16n\omega_n}\int_{R^n} \frac{u_k^2}{|x|^{2m+4}} dx \\
&\quad + \frac{\beta(W; R)}{n\omega_n}(\frac{n^2}{4})\int_{R^n} \frac{W(x)}{|x|^2}u_k^2 dx + \frac{\mu^2}{n\omega_n R^2}||u_k||_{W_0^{1,2}}.
\end{aligned}
$$

Hence (54) holds. $\qquad\square$

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